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Non-universal critical behaviour of two-dimensional Ising systems

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Abstract. Two conditions are derived for Ising models to show non-universal critical behaviour, namely, conditions concerning (i) logarithmic singularity of the specific heat and (ii) degeneracy of the ground state. These conditions are satisfied by the eight-vertex model, the Ashkin–Teller model, some Ising models with short- or long-range interactions and even Ising systems without the translational or rotational invariance.

1. Introduction

The universality of critical exponents is one of the most important concepts in critical phenomena. According to this universality hypothesis, critical exponents depend only upon the dimensionality, the symmetry and the interaction range of Hamiltonians, namely they are independent of the details of the relevant Hamiltonian such as the strength of the interactions in ordinary situations.

It is quite interesting to study under what condition this hypothesis is violated from the form of the Hamiltonian.

The first example that violates the universality hypothesis is the eight-vertex model solved by Baxter [1]. This model can be mapped [2, 3] onto the two-layered square-lattice Ising model, in which two layers interact with each other, via a four-body interaction J_4. The critical exponents of this model vary with parameter μ, which is a function of interaction energies. The exponents are obtained [1] as $\alpha = 2 - \pi/\mu$, $\beta = \pi/16\mu$, $\nu = \pi/2\mu$ and the scaling hypothesis or the weak universality hypothesis [4] insists that $\gamma = 7\nu/4 = 7\pi/8\mu$, $\delta = 15$, $\eta = 1/4$. Kadanoff and Wegner [3] have shown that the existence of marginal operators is a necessary condition for the appearance of continuously varying critical exponents. Kadanoff and Brown [5] have shown that the long-range behaviour of the correlations of the eight-vertex and the Ashkin–Teller models are asymptotically the same as those of the Gaussian model.

There exists another model which consists of short-range two-body interactions and which is believed to have continuously varying critical exponents. The $s = \frac{1}{2}$ square-lattice Ising model with the nearest-neighbour interaction J and the next-nearest-neighbour interaction J' have been studied by many authors [6–12] to obtain the phase diagram. The numerical calculations of van Leeuwen [13], Nightingale [14] and Swendsen and Krinsky [15] are the first to show the non-universal critical behaviour of this model. The singular part of the free energy is calculated perturbatively by Barber [16]. The results of the high-temperature expansion by Oitmaa [17] agree with these calculations. The coherent-anomaly method (CAM) [18–22] is applied to this model and the continuously varying critical exponents are estimated with errors smaller than $\sim 1\%$ [21, 22]. The degeneracy of the ground-state energy and the existence of a multi-component order parameter have been studied by Jüngling [23] and Krinsky and Muhamel [24]. A more general Hamiltonian has been investigated by the present authors [22] which includes the above two models as special cases and which has continuously varying critical exponents.

In the present paper, we phenomenologically derive a sufficient condition which has continuously varying critical exponents. This study is a generalization of the argument reported in [25]. Our condition is satisfied in the eight-vertex model, the Ashkin–Teller model and the $s = \frac{1}{2}$ square-lattice Ising model with next-nearest-neighbour interaction. Our condition is also satisfied by some systems including long-range interactions and with some systems without translational or rotational invariance.

Brief explanations of the relevant models are given in section 2, together with some explicit conditions on continuously varying critical exponents. A phenomenological perturbation scheme [26] is explained in section 3 and applied to the eight-vertex model in order to demonstrate that our scheme provides the exact first-order derivative of the critical temperature and exponent γ of the eight-vertex model in section 4. The temperature dependence of correlation functions and the characteristic cancellations of interaction energies at the ground state are discussed in section 5. The symmetry of the relevant model is studied in section 6. In section 7, the perturbational scheme explained in section 3 is applied to the model defined in section 2. It is derived that there exist finite first- or second-order derivatives (with respect to the interaction energies) of the critical exponent γ and, hence, that it varies continuously as a function of interaction energies. Finally, in section 8, the condition in section 2 is generalized to include a more general type of interaction.

2. Models

Let us consider the following Hamiltonian:

$$\mathcal{H} = \sum_k \mathcal{H}_k + \sum_{kl} \mathcal{H}_{kl} \tag{2.1}$$

where $\{\mathcal{H}_k\}$ is a finite set of two-dimensional Ising systems. In the present paper, we derive that the model with Hamiltonian (2.1) will have continuously varying critical exponents provided it satisfies the following two conditions.

(i) The specific heat of $\mathcal{H}_k$ shows the logarithmic singularity at the critical temperature T_c, which is independent of k.

(ii) The ground-state energy of the total Hamiltonian $\mathcal{H}$ is invariant for the spin inversion of each $\mathcal{H}_k$. Here, $\mathcal{H}_{kl}$ is written as

$$\mathcal{H}_{kl} = -J_{kl} \sum_i \mathcal{O}_i^{(k)} \mathcal{O}_i^{(l)} \tag{2.2}$$

where $\mathcal{O}_i^{(k)}$ and $\mathcal{O}_i^{(l)}$ are the $n^{(k)}$-body and $n^{(l)}$-body spin product of spins belonging to $\mathcal{H}_k$ and $\mathcal{H}_l$, respectively.

The logarithmic singularity in condition (i) results in the temperature dependence of the correlation functions of the form $\langle s_i s_j \rangle = c_0 + c_1 \epsilon \ln \epsilon$, where $\epsilon = (T - T_c)/T_c$ and c_0 and c_1 are some constants. This formula is derived in section 5.

The restriction on $\mathcal{O}_i^{(k)}$ and $\mathcal{O}_i^{(l)}$ in condition (ii) can be partially removed and more complicated interactions are permitted, i.e. $n^{(k)}$ and $n^{(l)}$ can depend on the region $\mathcal{R}_m$ in (5.6). This generalization is explained in section 8.

3. Perturbational scheme

To derive the non-universal critical behaviour of these models, let us consider the perturbational expansion [26] with respect to interaction energies. The susceptibility χ is assumed to behave as

$$\chi \sim \epsilon(J)^{-\gamma(J)} \qquad \epsilon(J) = (T - T_c(J))/T_c(J) \tag{3.1}$$

where J is an interaction energy. Differentiating χ with respect to J, we obtain

$$\left(\frac{\partial \chi}{\partial J}\right)_{J=0} \simeq \chi_0 \left[-\gamma(0)\frac{1}{\epsilon(0)}\left(\frac{\partial \epsilon}{\partial J}\right)_0 - \left(\frac{\partial \gamma}{\partial J}\right)_0 \log \epsilon(0)\right] \tag{3.2}$$

where the subscript 0 denotes $J = 0$. Hence, we can obtain $(\partial\gamma/\partial J)_0$ from the coefficient of the temperature dependence $\chi_0 \log \epsilon$. On the other hand, the susceptibility is expressed by the two-spin correlation functions in the form

$$\chi = \beta\mu_B^2 \sum_{i_0 j_0} g_{i_0 j_0} \langle s_{i_0} s_{j_0}\rangle \tag{3.3}$$

where $\beta = 1/k_B T$, μ_B is the Bohr magneton and $\{g_{i_0 j_0}\}$ denote the signs coming from the emerging order. We differentiate (3.3) and estimate $(\partial\gamma/\partial J)_0$ by comparison with (3.2). The existence of a finite and non-vanishing derivative of the exponent γ is evidence for the appearance of continuously varying critical exponents.

4. Example

Here we give two important examples. Let us consider the zero-field eight-vertex model. The Hamiltonian $\mathcal{H}_{8V}$ of this model is written as $\mathcal{H}_{8V} = \mathcal{H}_1 + \mathcal{H}_2 + \mathcal{H}_{12}$, where $\mathcal{H}_1$ and $\mathcal{H}_2$ denote the following Hamiltonians:

$$\begin{aligned} \mathcal{H}_1 &= -J' \sum_{i+j=\text{even}} (s_{ij}s_{i+1\,j+1} + s_{ij}s_{i+1\,j-1}) \\ \mathcal{H}_2 &= -J' \sum_{i+j=\text{odd}} (s_{ij}s_{i+1\,j+1} + s_{ij}s_{i+1\,j-1}) \end{aligned} \tag{4.1}$$

and $\mathcal{H}_{12}$ is

$$\mathcal{H}_{12} = -J_4 \sum s_{ij}s_{i+1\,j+1}s_{i+1\,j}s_{i\,j+1}. \tag{4.2}$$

This model decouples into the two square-lattice Ising models $\mathcal{H}_1$ and $\mathcal{H}_2$ when $J_4 = 0$. The interaction $\mathcal{H}_{12}$ has even symmetry for the spin inversion of each subsystem $\mathcal{H}_1$ and $\mathcal{H}_2$ and, hence, the energy of $\mathcal{H}_{8V}$ is four-fold degenerate in the whole temperature region. This model obviously satisfies conditions (i) and (ii) in section 2. The weight $g_{i_0 j_0}$ equals 1 for $J' > 0$. Differentiating (3.3) with respect to J_4 and using the fact that the model decouples into two layers when $J_4 = 0$, we obtain the following expression:

$$\left(\frac{\partial \chi}{\partial J_4}\right)_{J_4=0} = \beta^2\mu_B^2 \sum_{i_0 j_0 ijkl} \left[\langle s_{i_0 j_0} s_{ij} s_{kl} s_{k+1\,l+1}\rangle_0 - \langle s_{i_0 j_0} s_{ij}\rangle_0 \langle s_{kl} s_{k+1\,l+1}\rangle_0\right] \langle s_{k\,l+1} s_{k+1\,l}\rangle_0 \tag{4.3}$$

where $\langle\ \rangle_0$ denotes the expectation value for $J_4 = 0$. This expression coincides with

$$\left(\frac{\partial \chi}{\partial J'}\right)_{J_4=0} \omega_0 = \frac{\chi_0}{\epsilon(0)} \frac{\gamma(0)}{J'} \omega_0. \tag{4.4}$$

Here, ω_0 denotes the nearest-neighbour two-spin correlation function which shows the following behaviour:

$$\omega_0 \simeq \frac{1}{\sqrt{2}} + \frac{4J'}{\pi k_B T_c(0)} \epsilon(0) \log \epsilon(0) \tag{4.5}$$

at the critical point. Comparing (4.4) and (4.5) with (3.2), we obtain

$$\left(\frac{\partial T_c}{\partial J_4}\right)_{J_4=0} = \frac{T_c(0)}{\sqrt{2}J'} \qquad \text{and} \qquad \left(\frac{\partial \gamma}{\partial J_4}\right)_{J_4=0} = -\frac{4\gamma(0)}{\pi k_B T_c(0)} \tag{4.6}$$

which are identical to the derivatives obtained from the exact result by Baxter.

The square-lattice Ising model with antiferromagnetic next-nearest-neighbour interactions is another example. The Hamiltonian of this model is given in the form $\mathcal{H}_{SAF} = \mathcal{H}_1 + \mathcal{H}_2 + \mathcal{H}_{12}$, where $\mathcal{H}_1$ and $\mathcal{H}_2$ are given by (4.1) for $J' < 0$ and $\mathcal{H}_{12}$ is

$$\mathcal{H}_{12} = -J \sum_{ij} (s_{ij}s_{i+1j} + s_{ij}s_{ij+1}). \tag{4.7}$$

The ground state of this model is ordered as the Néel state in each sublattice for the interaction region $|J/J'| < 2$, in which the ground-state energy is invariant for the spin inversion of each sublattice. It is in this interaction region $|J/J'| < 2$ that this model is considered to have continuously varying critical exponents. This model also satisfies the condition in section 2. It is shown in [16, 25] that this model has continuously varying critical exponents with derivatives $(\partial\gamma/\partial J)_0 = 0$ and $(\partial^2\gamma/\partial J^2)_0 \neq 0$.

5. Preliminary formulae

The temperature dependence of correlation functions is specified from condition (i). For the purpose of obtaining multi-spin correlation functions, let us consider $\tilde{\mathcal{H}}_k = \mathcal{H}_k - \tilde{J}\mathcal{O}$, where $\mathcal{O}$ denotes a certain spin product. The free energy $\tilde{f}_{ks}$ of this Hamiltonian is differentiable with respect to $\tilde{J}$. Assuming that $\tilde{f}_{ks}$ shows logarithmic or power-law behaviour for $\tilde{J} = 0$, it should have the form

$$\tilde{f}_{ks} = C_1 \epsilon^{2-\alpha_1} \log \epsilon + C_2 \epsilon^2 \frac{1 - \epsilon^{-\alpha_2}}{\alpha_2} \tag{5.1}$$

where C_1 and C_2 are some constants, α_1 and α_2 are functions of $\tilde{J}$ with $\alpha_1(\tilde{J} = 0) = 0$ and $\alpha_2(\tilde{J}) \to 0$ for $\tilde{J} \to 0$. The second term converges to $C_2\epsilon^2 \log \epsilon$ when $\tilde{J} \to 0$. From (5.1),

$$\langle \mathcal{O} \rangle = \frac{1}{\beta} \left(\frac{\partial \tilde{f}_{ks}}{\partial \tilde{J}}\right)_{\tilde{J}=0} \propto 2\epsilon \left(\frac{\partial \epsilon}{\partial \tilde{J}}\right)_0 \log \epsilon + O(\epsilon) \tag{5.2}$$

where contributions vanishing faster than $\epsilon \log \epsilon$ (for $\epsilon \to 0$) are included in $O(\epsilon)$. The regular part of the free energy yields a certain constant term. Then, we generally obtain the temperature dependence of correlation functions in the form

$$\langle \mathcal{O} \rangle \cong c_0 + c_1 \epsilon \log \epsilon. \tag{5.3}$$

Condition (ii) results in the strong cancellation of interaction energies at the ground state. Here, we introduce notation for the ground-state configuration of the system as $\{g_i\}$ and the spin product $\mathcal{O}_i = s_{i_1} \cdots s_{i_n}$ in $\mathcal{H}_{kl}$ at the ground state as $\mathcal{G}_i = g_{i_1} \ldots g_{i_n}$. The J_{kl}-dependent part of the ground-state energy $\epsilon_G(J_{kl})$ is written as

$$\epsilon_G(J_{kl}) = -J_{kl} \sum_i \mathcal{G}_i. \tag{5.4}$$

In the case where $\sum_i \mathcal{O}_i$ has even symmetry for the spin inversion of $\mathcal{H}_k$ and $\mathcal{H}_l$ (i.e. both $n^{(k)}$ and $n^{(l)}$ are even), condition (ii) is automatically satisfied. Otherwise, condition (ii) yields

$$-J_{kl} \sum_i \mathcal{G}_i = -J_{kl} \sum_i (-\mathcal{G}_i) \qquad \text{that is,} \ \sum_i \mathcal{G}_i = 0. \tag{5.5}$$

We exclude from our arguments the case when condition (ii) is asymptotically satisfied only in the thermodynamic limit. Then, we can rewrite condition (5.5) as

$$\sum_i \mathcal{G}_i = \sum_m \sum_{i \in \mathcal{R}_m} \mathcal{G}_i \qquad \text{and} \qquad \sum_{i \in \mathcal{R}_m} \mathcal{G}_i = 0 \tag{5.6}$$

where $\{\mathcal{R}_m\}$ denote a set of finite regions containing a finite number of spins and $i \in \mathcal{R}_m$ denotes that all the spins in $\mathcal{G}_i$ are included in $\mathcal{R}_m$.

6. Symmetries and vanishing derivatives

6.1. Basic symmetries

In this section, we derive some properties which depend only on the symmetry of the relevant model. We consider here the case for $N = 2$, i.e. the Hamiltonian $\mathcal{H} = \mathcal{H}_1 + \mathcal{H}_2 + \mathcal{H}_{12}$ for $\epsilon > 0$.

We assume that $\mathcal{H}_{12}$ has odd symmetry for the spin inversion of $\mathcal{H}_2$. Then, the interaction $\mathcal{H}_{12}$ has the form

$$\mathcal{H}_{12} = -J \sum \mathcal{O}_i^{(1)} \mathcal{O}_i^{(2)} \tag{6.1}$$

where $\mathcal{O}_i^{(2)}$ changes its sign for the spin inversion of $\mathcal{H}_2$. Hereafter, we omit the subscript of J_{12} and write J_{12} as J for simplicity.

Let us consider the transformation of spin configurations in which all the spins on $\mathcal{H}_2$ are reversed and the spins on $\mathcal{H}_1$ are unchanged. The energy of model $\mathcal{H}$ with interaction J for spin configuration C equals the energy of model $\mathcal{H}$ with the interaction $-J$ for the transformed spin configuration C'. As a result, we can find one-to-one correspondence for the Boltzmann factor arising from configurations C and C'. Summing over all configurations, we find that the partition functions $Z(J)$ and $Z(-J)$ are the same

$$Z(J) = Z(-J). \tag{6.2}$$

Since we consider the partition function without an external field, we obtain from (6.2), for the critical temperature $T_c(J)$ and the exponent $\alpha(J)$ of the specific heat,

$$T_c(J) = T_c(-J) \qquad \text{and} \qquad \alpha(J) = \alpha(-J). \tag{6.3}$$

From (6.3), we can conclude that any odd derivatives of $T_c(J)$ and $\alpha(J)$ are equal to zero when they exist.

Next, we study the symmetry of the correlation functions $\omega_{i_0 j_0}(J)$, which are the expectation values of the following two-spin product with the weight corresponding to the emerging order as

$$\omega_{i_0 j_0}(J) \equiv \frac{\mathrm{Tr}\, g_{i_0 j_0} s_{i_0} s_{j_0} \exp[-\beta \mathcal{H}]}{Z(J)} \sim \exp[-r\epsilon(J)^{\nu(J)}] \tag{6.4}$$

and we also study the symmetry of the susceptibility

$$\chi(J) \equiv \sum_{i_0 j_0} \omega_{i_0 j_0}(J) \sim \epsilon(J)^{-\gamma(J)} \tag{6.5}$$

where $\{g_{i_0 j_0}\}$ is the sign corresponding to the emerging order and r is the distance between site i_0 and j_0. We can assume, without loss of generality, that site i_0 lies on $\mathcal{H}_1$. Corresponding to the change of interaction J to $-J$, the quantity $g_{i_0 j_0}$ changes its sign when site j_0 belongs to $\mathcal{H}_2$ and then $g_{i_0 j_0} s_{i_0} s_{j_0}$ is even for the spin inversion of $\mathcal{H}_2$. As a result, we again arrive at

$$\omega_{i_0 j_0}(J) = \omega_{i_0 j_0}(-J) \qquad \text{and} \qquad \chi(J) = \chi(-J) \tag{6.6}$$

and, hence,

$$\nu(J) = \nu(-J) \qquad \text{and} \qquad \gamma(J) = \gamma(-J). \tag{6.7}$$

6.2. *Vanishing cases*

The above argument cannot exclude the case when the critical temperature and exponents are not differentiable at $J = 0$. The critical coefficient, indeed, behaves as a cusp and is not differentiable at $J = 0$ when the system is not translationally invariant. Here, we show, using the scaling hypothesis, that the first-order derivatives of the critical temperature and exponents exist and they are vanishing. Let us consider the free energy f without the external field as

$$f \simeq C\epsilon^{2-\alpha} \log \epsilon \tag{6.8}$$

where C is some constant. Differentiating (6.8) with respect to interaction J and taking a limit $J \to 0$, we obtain

$$0 = \left(\frac{\partial C}{\partial J}\right)_0 \epsilon^2 \log \epsilon + C\epsilon^2 \frac{1}{\epsilon}\left(\frac{\partial \epsilon}{\partial J}\right)_0 + C\epsilon^2 \log \epsilon \left[(2-\alpha)\frac{1}{\epsilon}\left(\frac{\partial \epsilon}{\partial J}\right)_0 - \left(\frac{\partial \alpha}{\partial J}\right)_0 \log \epsilon\right] \tag{6.9}$$

and, as a result,

$$\left(\frac{\partial C}{\partial J}\right)_0 = 0 \qquad \left(\frac{\partial \alpha}{\partial J}\right)_0 = 0 \qquad \text{and} \qquad \left(\frac{\partial \epsilon}{\partial J}\right)_0 = 0. \tag{6.10}$$

Next, let us consider the free energy f with external field h. We assume the scaling form of f as

$$f_J(\lambda^p\epsilon, \lambda^q h) = \lambda f_J(\epsilon, h) \tag{6.11}$$

where λ is some parameter and p and q are numbers independent of ϵ and h. They are related to exponents α and γ by

$$\alpha = 2 - \frac{1}{p} \qquad \text{and} \qquad \gamma = \frac{2q-1}{p}. \tag{6.12}$$

From (6.10) and (6.12), we obtain $(\partial p/\partial J)_0 = 0$. The partial derivative of f shows the same scaling form as

$$\frac{\partial f_J(\lambda^p\epsilon, \lambda^q h)}{\partial J} = \lambda \frac{\partial f_J(\epsilon, h)}{\partial J}. \tag{6.13}$$

Differentiating (6.11) with respect to J, a straightforward calculation yields

$$M(0,1)\frac{\log h}{q}\left(\frac{\partial q}{\partial J}\right)_{J=0} = 0 \tag{6.14}$$

for $\epsilon \to 0$, $J \to 0$ and $\lambda^q h = 1$, where $M(\epsilon, h) = \partial f_J(\epsilon, h)/\partial h$ is the magnetization. Note that $(\partial q/\partial J)$ is independent of h. From (6.14), we obtain $(\partial q/\partial J)_{J=0} = 0$ and, as a result,

$$\left(\frac{\partial \gamma}{\partial J}\right)_{J=0} = 0. \tag{6.15}$$

7. The non-vanishing derivatives

In this section, we show the non-universal behaviour of the relevant system when it satisfies conditions (i) and (ii). We use the formulae (5.3) and (5.6) derived from (i) and (ii), respectively, and (6.10) and (6.15) result from the symmetry of the model. The properties (5.6), (6.10) and (6.15) are valid when $n^{(k)}$ or $n^{(l)}$ (or both $n^{(k)}$ and $n^{(l)}$) are odd. Let us consider the weighted susceptibility

$$\chi = \beta\mu_{\rm B}^2 \sum_{i_0 j_0} g_{i_0 j_0}\langle s_{i_0} s_{j_0}\rangle \tag{7.1}$$

where $g_{i_0 j_0}$ is the sign corresponding to the emerging order as $g_{i_0 j_0} = \mathrm{sgn}(g_{i_0} g_{j_0})$. All we have to do here is to differentiate (7.1) in terms of J_{kl} and show that the second dominant term shows the logarithmic singularity $\chi_0 \log\epsilon$ with $\epsilon = (T - T_c)/T_c$.

Let us introduce the following notation. Interaction $\mathcal{H}_{kl}$ is expressed as

$$\mathcal{H}_{kl} = -J\sum_i \mathcal{O}_i \qquad \text{and} \qquad \mathcal{O}_i = \mathcal{O}_i^{(k)}\mathcal{O}_i^{(l)} \tag{7.2}$$

where J_{kl} is written as J for simplicity. The vector r_i denotes the coordinate of the spin s_i and $R_i = \{r_{i_1}, \ldots, r_{i_n}\}$ denotes the coordinate of the spin product $\mathcal{O}_i = s_{i_1}\cdots s_{i_n}$. The expectation value of $\mathcal{O}_i$ is expressed as a function of R_i as

$$\langle \mathcal{O}_i \rangle = c_0(R_i) + c_1(R_i)\epsilon\log\epsilon. \tag{7.3}$$

We also write $R_i \subset \mathcal{R}_m$ (or $i \in \mathcal{R}_m$) when $r_s \in \mathcal{R}_m$ for all $r_s \in R_i$.

Here we derive finite derivatives of exponent γ. The first-order derivative of (7.1) is

$$\left(\frac{\partial \chi}{\partial J}\right)_{J=0} = \beta^2\mu_{\rm B}^2 \sum_{i_0 j_0} g_{i_0 j_0} \sum_i \left[\langle s_{i_0} s_{j_0} \mathcal{O}_i^{(k)}\mathcal{O}_i^{(l)}\rangle_0 - \langle s_{i_0} s_{j_0}\rangle_0 \langle \mathcal{O}_i^{(k)}\mathcal{O}_i^{(l)}\rangle_0\right] \tag{7.4}$$

where $\langle\ \rangle_0$ denotes the expectation value taken for $J = 0$. All the terms cancel except the following two cases:

(I-1) $s_{i_0}, s_{j_0} \in \mathcal{H}_k$ (or $s_{i_0}, s_{j_0} \in \mathcal{H}_l$) and both $n^{(k)}$ and $n^{(l)}$ are even; and
(I-2) $s_{i_0} \in \mathcal{H}_k$, $s_{j_0} \in \mathcal{H}_l$ and both $n^{(k)}$ and $n^{(l)}$ are odd.

The derivatives $(\partial\gamma/\partial J)_0$ and $(\partial T_c/\partial J)_0$ of the latter case (I-2) vanish as shown in section 6. As a result, we have only to treat the case (I-1) as the first-order derivative. This case is a simple generalization of the argument for the eight-vertex model, as in section 4. Derivative (7.4) is written as

$$\left(\frac{\partial\chi}{\partial J}\right)_{J=0} = \beta^2\mu_B^2\sum_{i_0j_0} g_{i_0j_0}\sum_i\left[\langle s_{i_0}s_{j_0}\mathcal{O}_i^{(k)}\rangle_0 - \langle s_{i_0}s_{j_0}\rangle\langle\mathcal{O}_i^{(k)}\rangle_0\right]\langle\mathcal{O}_i^{(l)}\rangle_0. \quad (7.5)$$

This is a generalization of (4.3). From (3.2) and (5.3), we obtain the following temperature dependence:

$$\left(\frac{\partial\chi}{\partial J}\right)_{J=0} \simeq \chi_0\sum_i \mathcal{G}_i\left[a_0(R_i)\frac{1}{\epsilon} + a_1(R_i)\log\epsilon\right][c_0(R_i) + c_1(R_i)\epsilon\log\epsilon] \quad (7.6)$$

where $a_0(R_i)$, $a_1(R_i)$, $c_0(R_i)$ and $c_1(R_i)$ are some constants. The term $\mathcal{G}_i = \mathcal{G}_i^{(k)}\mathcal{G}_i^{(l)}$ is factorized so that the inside of the brackets $[\cdots][\cdots]$ in (7.6) are positive. We generally cannot omit the term $a_1(R_i)$. The sums $\sum_i \mathcal{G}_i^{(k)}a_0(R_i)$ and $\sum_i \mathcal{G}_i^{(k)}a_1(R_i)$ appear as the coefficient of $(\partial\tilde{\chi}_k/\partial\tilde{J}_k)_{\tilde{J}_k=0}$, where $\tilde{\chi}_k$ is the susceptibility of the model described by the Hamiltonian $\tilde{\mathcal{H}}_k = \mathcal{H}_k - \tilde{J}_k\sum_i \mathcal{O}_i^{(k)}$, and they are finite in the thermodynamic limit. The coefficient of $\log\epsilon$, namely

$$\sum_i \mathcal{G}_i(a_0(R_i)c_1(R_i) + a_1(R_i)c_0(R_i)) \quad (7.7)$$

is finite because $\sum_i \mathcal{G}_i^{(k)}a_0(R_i)$, $\sum_i \mathcal{G}_i^{(k)}a_1(R_i)$, $\mathcal{G}_i^{(l)}c_0(R_i)$ and $\mathcal{G}_i^{(l)}c_1(R_i)$ are all finite. This is the first-order derivative of γ.

For all cases, except when both $\mathcal{O}_i^{(k)}$ and $\mathcal{O}_i^{(l)}$ have even symmetry (i.e. case (I-1): both $n^{(k)}$ and $n^{(l)}$ are even), we obtain, from (6.10) and (6.15),

$$\left(\frac{\partial T_c}{\partial J}\right)_0 = 0 \qquad \text{and} \qquad \left(\frac{\partial\gamma}{\partial J}\right)_0 = 0. \quad (7.8)$$

In these cases, we have to show that the second-order derivatives are non-vanishing and finite. From (3.1) and (7.8), the second-order derivative of the susceptibility χ is

$$\left(\frac{\partial^2\chi}{\partial J^2}\right)_{J=0} \simeq \chi_0\left[-\gamma(0)\frac{1}{\epsilon(0)}\left(\frac{\partial^2\epsilon}{\partial J^2}\right)_0 - \left(\frac{\partial^2\gamma}{\partial J^2}\right)_0\log\epsilon(0)\right]. \quad (7.9)$$

Hence, the existence of the logarithmic singularity is the sign of continuously varying critical exponents. All we have to do is find a term proportional to $\chi_0\log\epsilon$ in the second-order derivative of χ and to show that the coefficient of $\chi_0\log\epsilon$ is finite.

Differentiating $\chi = \beta\mu_B^2\sum\langle s_{i_0}s_{j_0}\rangle$ twice with respect to J, we obtain

$$\left(\frac{\partial^2\chi}{\partial J^2}\right)_{J=0} = \beta^3\mu_B^2\sum_{i_0j_0} g_{i_0j_0}\sum_i\sum_j\left[\langle s_{i_0}s_{j_0}\mathcal{O}_i^{(k)}\mathcal{O}_j^{(k)}\rangle_0 - \langle s_{i_0}s_{j_0}\rangle_0\langle\mathcal{O}_i^{(k)}\mathcal{O}_j^{(k)}\rangle_0\right]\langle\mathcal{O}_i^{(l)}\mathcal{O}_j^{(l)}\rangle_0 \quad (7.10)$$

where we have used the fact that $n^{(k)}$ or $n^{(l)}$ (or both $n^{(k)}$ and $n^{(l)}$) are odd and that the expectation value $\langle s_{i_1}\ldots s_{i_n}\rangle_0$ equals zero when n is odd. The following case remains non-vanishing:

(II-1) $s_{i_0}, s_{j_0} \in \mathcal{H}_k$ (or $s_{i_0}, s_{j_0} \in \mathcal{H}_l$) $n^{(k)}$ is even and $n^{(l)}$ is odd; and
(II-2) $s_{i_0}, s_{j_0} \in \mathcal{H}_k$ (or $s_{i_0}, s_{j_0} \in \mathcal{H}_l$) and both $n^{(k)}$ and $n^{(l)}$ are odd.

Both cases can be treated simultaneously. From (3.2) and (5.3), derivative (7.10) shows the following temperature dependence:

$$\chi_0 \sum_i \sum_j \mathcal{G}_i \mathcal{G}_j \left[a_0(R_i, R_j)\frac{1}{\epsilon} + a_1(R_i, R_j)\log\epsilon \right][c_0(R_i, R_j) + c_1(R_i, R_j)\epsilon\log\epsilon]$$
$$\equiv \chi_0 \sum_i \sum_j \mathcal{G}_i \mathcal{G}_j \Omega(R_i, R_j) \tag{7.11}$$

where $\Omega(R_i, R_j)$ is positive and $a_0(R_i, R_j)$, $a_1(R_i, R_j)$, $c_0(R_i, R_j)$ and $c_1(R_i, R_j)$ are some constants. This summation can be regrouped by $\{\mathcal{R}_m\}$, and using condition (56) (i.e. condition (ii): $\sum_{i\in\mathcal{R}_m} \mathcal{G}_i = 0$), we obtain

$$\chi_0 \sum_{mm'} \sum_{i\in\mathcal{R}_m} \sum_{j\in\mathcal{R}_{m'}} \mathcal{G}_i \mathcal{G}_j \Omega(R_m + \Delta R_{mi}, R_{m'} + \Delta R_{m'j})$$
$$\simeq \chi_0 \sum_{mm'} \sum_{i\in\mathcal{R}_m j\in\mathcal{R}_{m'}} \mathcal{G}_i \mathcal{G}_j (\Delta R_{mi}\cdot\nabla)(\Delta R_{m'j}\cdot\nabla')\Omega(R_m, R_{m'}) \tag{7.12}$$

where $R_m \subset \mathcal{R}_m$ and $R_{m'} \subset \mathcal{R}_{m'}$ are fixed for each $\mathcal{R}_m$ and $\mathcal{R}_{m'}$, respectively, and we have used the following notation:

$$\Delta R_{mi}\cdot\nabla \equiv \Delta r_{m_1 i_1}\cdot\nabla_{i_1} + \cdots + \Delta r_{m_n i_n}\cdot\nabla_{i_n}$$
$$\Delta R_{m'j}\cdot\nabla' \equiv \Delta r_{m'_1 j_1}\cdot\nabla'_{j_1} + \cdots + \Delta r_{m'_n j_n}\cdot\nabla'_{j_n} \tag{7.13}$$

where ∇_i and ∇'_i are the gradients operating on the coordinates of s_i and $\Delta_{mi}\cdot\nabla$, and $\Delta_{m'j}\cdot\nabla'$ operates on the first and second arguments of $\Omega(R_m, R_{m'})$.

The first summation in (7.12) is classified by the distance between $\mathcal{R}_m$ and $\mathcal{R}_{m'}$ as

$$\sum_{mm'} = \sum_r \sum_{mm'|m-m'|=r} \tag{7.14}$$

where $|m-m'| = r$ denotes that $r \leqslant \min\{|r_i - r_{i'}| \mid r_i \in \mathcal{R}_m, r_{i'} \in \mathcal{R}_{m'}\} < r + \Delta r$ and Δr is a constant comparable to the mean size of $\{\mathcal{R}_m\}$.

As Ω is a smooth and decreasing function of r and all $\{\mathcal{R}_m\}$ denote finite regions containing a finite number of spins, each term in (7.12) is bounded by

$$\mathcal{G}_i \mathcal{G}_j (\Delta R_{mi}\cdot\nabla)(\Delta R_{m'j}\cdot\nabla')\Omega \leqslant \text{const}(\Delta R\cdot\nabla)(\Delta R\cdot\nabla')\Omega \tag{7.15}$$

where

$$\Delta R\cdot\nabla \equiv \Delta r\cdot\nabla_{i_1} + \cdots + \Delta r\cdot\nabla_{i_n} \tag{7.16}$$

and a similar equation is defined for $\Delta R\cdot\nabla'$.

Then, the coefficient of $\chi_0\log\epsilon$ in (7.12) is bounded as

$$\sum_r (\Delta R_{mi}\cdot\nabla)(\Delta R_{m'j}\cdot\nabla')F(r, \{R_i\}) < \sum_{r<r_0} (\Delta R_{mi}\cdot\nabla)(\Delta R_{m'j}\cdot\nabla')F(r, \{R_i\})$$
$$+ \text{const}\int_{r_0}^{\infty} \mathrm{d}^2 r\, (\Delta R\cdot\nabla)(\Delta R\cdot\nabla')F(r, \{R_m\}) \tag{7.17}$$

where

$$F(r,\{R_i\}) = \sum_{mm'|m-m'|=r} \sum_{i\in\mathcal{R}_m j\in\mathcal{R}_{m'}} \mathcal{G}_i\mathcal{G}_j[a_0(R_i,R_j)c_1(R_i,R_j)+a_1(R_i,R_j)c_0(R_i,R_j)] \tag{7.18}$$

is a finite function of r (see (7.7)).

Our aim is to show that (7.17) is finite. The first sum is, of course, finite and the second term is bounded by

$$A\int_{r_0}^{\infty} \mathrm{d}^2r\,\nabla^2 F(r,\{R_i\}) = A\int_{r=r_0} \mathrm{d}S\,\frac{\partial F}{\partial r} \simeq A\cdot 2\pi r_0 \frac{\partial F}{\partial r}\bigg|_{r=r_0} < \infty \tag{7.19}$$

where A is some constant.

8. Generalization

Condition (ii) can be generalized in the following two points.

First, it is straightforward to generalize the form of the interaction $\mathcal{H}_{kl}$ as

$$\mathcal{H}_{kl} = \sum_p \mathcal{H}_{kl}^{(p)} \qquad \text{and} \qquad \mathcal{H}_{kl}^{(p)} = -J_{kl}^{(p)} \sum \mathcal{O}_{pi}^{(k)}\mathcal{O}_{pi}^{(l)} \tag{8.1}$$

where each $\mathcal{H}_{kl}^{(p)}$ satisfies condition (ii).

Second, the values of $n^{(k)}$ and $n^{(l)}$ have been fixed. This condition is necessary for showing the cancellations of correlations in the zeroth- and first-order terms in (7.12) using the condition $\sum_{i\in\mathcal{R}_m}\mathcal{G}_i = 0$. However, our argument is valid for more complicated interactions, i.e. the case when $n^{(k)}$ and $n^{(l)}$ depend on the region $\mathcal{R}_m$. Each contribution from each $\mathcal{R}_m$ to the derivatives is classified into the cases shown in sections 6 and 7. For the first-order derivatives, there is no difference. For the second-order derivative, all the contributions to $(\partial^2\gamma/\partial J^2)_0$ coming from $\mathcal{R}_m$ and $\mathcal{R}_{m'}$ vanish except when both $n_m^{(k)} + n_{m'}^{(k)}$ and $n_m^{(l)} + n_{m'}^{(l)}$ are even. (Otherwise the argument in section 6 is valid for each term and the contributions to the derivative vanish.) The non-vanishing case can be treated in the same way as in section 7.

9. Conclusion

We have proved that some Ising systems satisfying conditions (i) and (ii) in section 2 show non-universal critical behaviour. The perturbational expansion in terms of the interaction J is performed. This method results in the exact first-order derivatives (4.6) in the case of the eight-vertex model. We have used conditions (5.3) and (5.6), which are derived from conditions (i) and (ii), respectively, and (6.10) and (6.15), which result from the symmetry of the model. The existence of finite and non-zero derivatives $(\partial\gamma/\partial J)_0$ or $(\partial^2\gamma/\partial J^2)_0$ is evidence of continuous variation of critical exponents. These derivatives are derived in (7.4)–(7.7) and (7.10)–(7.19) for the first- and the second-order derivatives, respectively. Finally, the straightforward generalization of condition (ii) is commented on in section 8. This condition is valid for generalized spin-S Ising models and can easily be generalized for other classical systems.

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